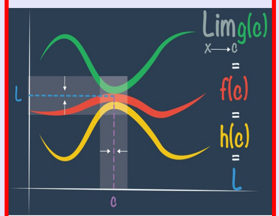
Calculus I

Lecture 7



Feb 19-8:47 AM

Class QZ 8

Use ε and δ to prove $\lim_{x\to 3}(8-2x)=2$.

$f(x)=8-2x$

$a=3$

$L=2$ ✓

For every $\varepsilon>0$, there is a $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|8-2x-2|<\varepsilon$ whenever $|x-3|<\delta$

$|6-2x|<\varepsilon$ " $|x-3|<\delta$

$|-2(x-3)|<\varepsilon$

$|-2|\,|x-3|<\varepsilon$

$2|x-3|<\varepsilon$ → Divide by 2

$|x-3|<\frac{\varepsilon}{2}$

$\boxed{\text{Pick } \delta=\frac{\varepsilon}{2}}$

Jan 14-7:59 AM

Use def. of derivative to find $f'(x)$ for $f(x)=x^2-8x-3$, then Solve $f'(x)=0$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-8(x+h)-3\overbrace{}^{-x^2+8x+3}}{h}$$

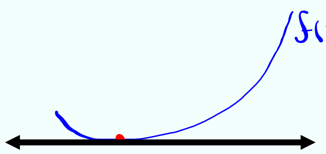
$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2\cancel{-8x}-8h\cancel{-3}\cancel{-x^2}\cancel{+8x}\cancel{+3}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+\cancel{h}^{\to 0}-8)}{\cancel{h}}=\boxed{2x-8}$$

$f'(x)=0$

$2x-8=0$

$\boxed{x=4}$



$f(x)=x^2-8x-3$

$(4,f(4))$

horizontal tan. line

$f'(4)=0$

Jan 14-8:17 AM

find $f'(x)$

$f(t)=t^3-5t^2$

$f'(t)=3t^2-10t$

1) $f(\underset{\uparrow}{x})=\underbrace{t^3-5t^2}_{\text{constant}}$ $\qquad \boxed{f'(x)=0}$

variable

2) $f(x)=\pi$ $\qquad \boxed{f'(x)=0}$

3) $f(x)=\underbrace{\sin^2 x+\cos^2 x}$

$f(x)=1$ $\qquad \boxed{f'(x)=0}$

constant

Jan 14-8:23 AM

Find $f'(x)$, then Solve $f'(x)=0$

1) $f(x) = 4x + 5$ $\quad f'(x) = 4 + 0$ $\quad f'(x) = 4$

Solve $f'(x)=0$

No Soln.

2) $f(x) = \frac{1}{2}x^2 - 4x - 1$ $\quad f'(x) = \frac{1}{2}\cdot 2x - 4 - 0$

$f'(x) = x - 4$

$f'(x) = 0$

$x - 4 = 0 \rightarrow \boxed{x=4}$

3) $f(x) = \frac{3}{4}x^8$

$f'(x) = \frac{3}{\cancel{4}}\cdot \overset{2}{\cancel{8}}\cdot x^{8-1}$

$= 6x^7$

$f'(x) = 0$

$6x^7 = 0$

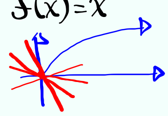
$\boxed{x=0}$

Jan 14-8:29 AM

Find $f'(x)$, then give x-values where $f'(x)=0$ or $f'(x)$ is undefined.

1) $f(x) = \sqrt{x}$ $\quad f(x) = x^{1/2}$ $\quad f'(x) = \frac{1}{2}x^{\frac{1}{2}-1}$

$x \geq 0$



$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$

$f'(x) = \frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2x}$

$f'(x) \neq 0$

$f'(x)$ is undefined at $\boxed{x=0}$

2) $f(x) = \frac{x}{x^2-1}$

$f'(x) = \frac{1(x^2-1) - x\cdot 2x}{(x^2-1)^2}$

$f'(x) = \frac{x^2 - 1 - 2x^2}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2}$

$f'(x) = 0 \rightarrow -x^2 - 1 = 0$ $\quad -x^2 = 1$ $\quad x^2 = -1$

No real Soln.

$f'(x)$ is undefined $\rightarrow (x^2-1)^2 = 0$

$x^2 - 1 = 0$

$x^2 = 1 \rightarrow \boxed{x = \pm 1}$

Jan 14-8:37 AM

$f(x) = \frac{x}{x-4} \rightarrow$ V.A. $x=4$

1) Domain $x \neq 4$

$(-\infty,4)\cup(4,\infty)$

2) x-Int $y=0$ $f(x)=0$

$\frac{x}{x-4}=0$ $x=0$

$(0,0)$

3) y-Int $x=0$

$y=\frac{0}{0-4}=0$

$(0,0)$

4) Find $f'(x)$

$f'(x) = \frac{1(x-4)-x\cdot 1}{(x-4)^2}$

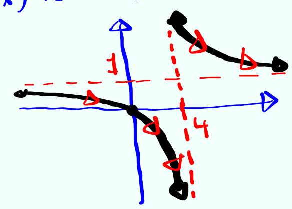
$= \frac{x-4-x}{(x-4)^2} = \frac{-4}{(x-4)^2}$

Sign chart

	$x=2$		$x=5$	
$-\infty$		4		∞
$f'(x)$	$-$	0	$-$	
	↘		↘	

$f'(x) \neq 0$

$f'(x)$ is und. at $x=4$



Jan 14-8:49 AM

$f(x) = x^2 - 6x$ Polynomial $\rightarrow$ Cont. $(-\infty,\infty)$

1) Domain

$(-\infty,\infty)$

2) x-Ints $y=0$

$f(x)=0$

$x^2-6x=0$

$x(x-6)=0$

$x=0$ $x=6$

$(0,0)$

$(6,0)$

3) Y-Int $x=0$

$y=0^2-6(0)=0$

$(0,0)$

4) $f'(x)$

$=2x-6$

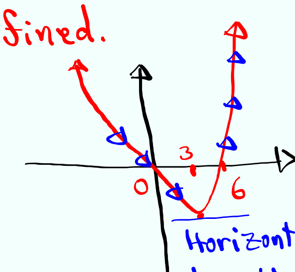
5) Find x-values where $f'(x)=0$ or Undefined.

$f'(x)=0$ $2x-6=0$ $\boxed{x=3}$

$f'(x)$ is never undefined.

Sign Chart

	$x=0$		$x=4$	
$-\infty$		3		∞
$f'(x)$	$-$	●	$+$	
	↘		↗	



Horizontal tan. line at $x=3$

Jan 14-9:01 AM

Find all $\boxed{\text{points}}$ on the graph of $y = x^4 + 2x^2 - 3$ where we have $\underbrace{\text{horizontal tan. lines.}}_{y' = 0}$

$y' = 4x^3 + 4x$

$4x^3 + 4x = 0$

$4x(x^2 + 1) = 0$

$4x = 0$ $\quad$ $x^2 + 1 \neq 0$

$\boxed{x = 0}$

$y = 0^4 + 2(0)^2 - 3 = \boxed{-3}$

Point $(0, -3)$

Jan 14-9:12 AM

Recall $\lim_{x \to 0} \frac{\sin x}{x} = 1$ & $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

Evaluate

$\lim_{x \to 0} \frac{\sin x}{x^2 + 5x} = \frac{0}{0}$ I.F.

$= \lim_{x \to 0} \frac{\sin x}{x(x+5)}$

$= \underbrace{\lim_{x \to 0} \frac{\sin x}{x}}_{\to 1} \cdot \underbrace{\lim_{x \to 0} \frac{1}{x+5}}_{\to \frac{1}{5}}$

$= 1 \cdot \frac{1}{5} = \boxed{\frac{1}{5}}$

$\lim_{x \to 2} \frac{\cos(x-2) - 1}{x-2}$

Let $t = x - 2$

$x \to 2$, $t \to 0$

$= \lim_{t \to 0} \frac{\cos t - 1}{t}$

$= \boxed{0}$

Jan 14-9:19 AM

1) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{\sin 8x} = \dfrac{0}{0}$ I.F.

$= \lim\limits_{x\to 0} \dfrac{\dfrac{5\sin 5x}{5x}}{\dfrac{8\sin 8x}{8x}}$

$= \dfrac{5\cdot \underbrace{\lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}}_{\to 1}}{8\cdot \underbrace{\lim\limits_{x\to 0}\dfrac{\sin 8x}{8x}}_{\to 1}} = \boxed{\dfrac{5}{8}}$

2) $\lim\limits_{x\to 1} \dfrac{\sin(x-1)}{x^2+x-2} = \dfrac{0}{0}$ I.F.

$= \lim\limits_{x\to 1} \dfrac{\sin(x-1)}{(x-1)(x+2)}$

$= \underbrace{\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{x-1}}_{\to 1} \cdot \underbrace{\lim\limits_{x\to 1}\dfrac{1}{x+2}}$

$= 1\cdot \dfrac{1}{1+2} = \boxed{\dfrac{1}{3}}$

Jan 14-9:28 AM

Derivatives of Trig. Functions:

$\dfrac{d}{dx}[\sin x] = \cos x$

$\dfrac{d}{dx}[\csc x] = -\csc x \cot x$

$\dfrac{d}{dx}[\cos x] = -\sin x$

$\dfrac{d}{dx}[\sec x] = \sec x \tan x$

$\dfrac{d}{dx}[\tan x] = \sec^2 x$

$\dfrac{d}{dx}[\cot x] = -\csc^2 x$

$$\boxed{\begin{aligned} y &= x^2 + \sin x \\ y' &= 2x + \cos x \end{aligned}}$$

$y = x^2\cos x$

$y' = 2x\cdot\cos x + x^2\cdot -\sin x$

$\boxed{y' = 2x\cos x - x^2\sin x}$

Jan 14-10:00 AM

$$f(x) = \frac{\sin x}{1+\cos x}$$

$$f'(x) = \frac{\cos x(1+\cos x) - \sin x\cdot(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2}$$

$$= \frac{\cancel{1+\cos x}}{(1+\cos x)^{\cancel{2}}} = \boxed{\frac{1}{1+\cos x}}$$

Jan 14-10:06 AM

find $f'(x)$

1) $f(x) = 3x^2 - 2\cos x$

$f'(x) = 3\cdot 2x - 2\cdot(-\sin x)$ $\boxed{f'(x) = 6x + 2\sin x}$

2) $f(x) = (3x^2-2)\cos x$

Product

$$\boxed{\begin{aligned} f'(x) &= 6x\cdot\cos x + (3x^2-2)\cdot(-\sin x) \\ &= 6x\cos x - (3x^2-2)\cdot\sin x \end{aligned}}$$

Jan 14-10:10 AM

3) $f(x) = \sin x + \cos x$

$\boxed{f'(x) = \cos x - \sin x}$

4) $f(x) = \sin x \cdot \cos x$

$f'(x) = \cos x \cdot \cos x + \sin x \cdot -\sin x$

$= \cos^2 x - \sin^2 x$

$= \boxed{\cos 2x}$

5) $f(x) = \dfrac{\sin x}{\cos x}$

$f(x) = \tan x$

$\dfrac{d}{dx}\left[\tan x\right] = \sec^2 x$

$f'(x) = \dfrac{\cos x \cdot \cos x - \sin x \cdot -\sin x}{\cos^2 x}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \dfrac{1}{\cos^2 x}$

$= \sec^2 x$

Jan 14-10:16 AM

find $f'(x)$ for

1) $f(x) = \cos x \cdot (1 - \sin x)$

$f'(x) = -\sin x\,(1 - \sin x) + \cos x \cdot (-\cos x)$

$= -\sin x + \sin^2 x - \cos^2 x = -\sin x - (\cos^2 x - \sin^2 x)$

$= \boxed{-\sin x - \cos 2x}$

2) $f(x) = \dfrac{\cos x}{1 - \sin x}$

$f'(x) = \dfrac{-\sin x(1 - \sin x) - \cos x(-\cos x)}{(1 - \sin x)^2}$

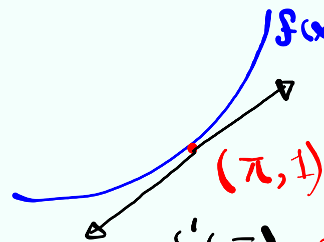
$= \dfrac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} = \dfrac{\cancel{1 - \sin x}}{(1 - \sin x)^{\cancel{2}}}$

$= \boxed{\dfrac{1}{1 - \sin x}}$

Jan 14-10:24 AM

Find equation of the tan. line to the graph of $f(x) = \sin x - \cos x$ at $x = \pi$.

$f(x) = \sin x - \cos x$



$f(\pi) = \sin\pi - \cos\pi$
$= 0 - (-1)$
$= 1$

$m = f'(\pi) = \cos\pi + \sin\pi$
$= -1 + 0$
$= \boxed{-1}$

$f(x) = \sin x - \cos x$

$f'(x) = \cos x - (-\sin x)$

$f'(x) = \cos x + \sin x$

$y - y_1 = m(x - x_1)$

$y - 1 = -1(x - \pi)$

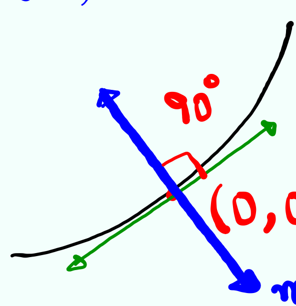
$\boxed{y = -x + \pi + 1}$

Jan 14-10:36 AM

Find eqn of the normal line to the graph of $f(x) = x + \tan x$ at $x = 0$.

$f(x) = x + \tan x$

$f(0) = 0 + \tan 0$
$= 0$



$m = \dfrac{-1}{f'(0)} = \boxed{\dfrac{-1}{2}}$

$f(x) = x + \tan x$

$f'(x) = 1 + \sec^2 x$

$f'(0) = 1 + \sec^2 0$

$= 1 + \dfrac{1}{\cos^2 0} = 1 + \dfrac{1}{1^2} = 2$

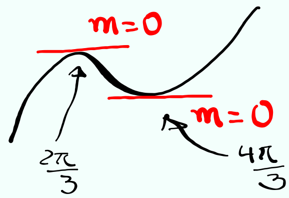
$y - y_1 = m(x - x_1)$

$y - 0 = \frac{-1}{2}(x - 0)$

$\boxed{y = \frac{-1}{2}x}$

Jan 14-10:42 AM

Find x-values in the interval $[0, 2\pi]$ where $f(x) = x + 2\sin x$ has horizontal tan. line.



$f'(x) = 0$

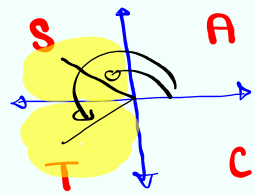
$f'(x) = 1 + 2 \cdot \cos x$

Solve $1 + 2\cos x = 0$ $\quad \cos x = \frac{-1}{2}$

Cos of what is $\frac{1}{2}$?

60° $\quad \frac{\pi}{3}$

Ref. Angle



QII $\rightarrow$ $\pi - RA = \pi - \frac{\pi}{3} = \boxed{\frac{2\pi}{3}}$

QIII $\rightarrow$ $\pi + RA = \pi + \frac{\pi}{3} = \boxed{\frac{4\pi}{3}}$

Jan 14-10:48 AM

For $\varepsilon = .2$, find a $\boxed{0 < \delta \le 2}$ such that $\lim_{x \to 5} (x^2 + 5) = 30$.

$f(x) = x^2 + 5$

$a = 5$

$L = 30$ ✓

For every $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 + 5 - 30| < \varepsilon$ " $|x - 5| < \delta$

$|x^2 - 25| < \varepsilon$ " $|x - 5| < \delta$

$|(x+5)(x-5)| < \varepsilon$

$|x+5|\,|x-5| < \varepsilon$

if $|x+5| = C$, then $C|x-5| < \varepsilon$ $\quad |x-5| < \frac{\varepsilon}{C}$

we wish $0 < \delta \le 2$ $\quad \rightarrow |x+5| < 12$

$|x - 5| < 2$

$-2 < x - 5 < 2$

Add 10

$8 < x + 5 < 12$

$\delta = \min\left\{2, \frac{\varepsilon}{12}\right\}$

For $\varepsilon = .2$

$\delta = \min\left\{2, \frac{.2}{12}\right\}$

$= \min\left\{2, \frac{1}{60}\right\}$

$\boxed{\text{Choose } \delta = \frac{1}{60}}$

Jan 14-10:56 AM

For $\varepsilon>0$, Find a $\delta>0$ such that

$\lim_{x\to 4}(x^2+5x)=36.$

$f(x)=x^2+5x$

$a=4$

$L=36$ ✓

For every $\varepsilon>0$, there is a $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|x^2+5x-36|<\varepsilon$ " $|x-4|<\delta$

$\underbrace{|x+9|}_{\text{Bound}}\underbrace{|x-4|}_{\text{Keep}}<\varepsilon$

If $|x+9|<C$, then $|x+9||x-4|<C|x-4|<\varepsilon$

$|x-4|<\frac{\varepsilon}{C}$

For $\delta\le 1$

$|x-4|<1$

$-1<x-4<1$

$+13 \quad +13 \quad +13$

$|x+9|<14$

$\delta=\min\left\{1,\frac{\varepsilon}{14}\right\}$

Jan 14-11:06 AM

Evaluate $\lim_{x\to\frac{\pi}{6}}\frac{\sin x-\frac{1}{2}}{x-\frac{\pi}{6}}$

$f(a)=$

$f(\frac{\pi}{6})=$

$\sin\frac{\pi}{6}$

Hint:

$f(x)=\sin x$

$a=\frac{\pi}{6}$

$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$

$f'(x)=\cos x$

$=\cos\left(\frac{\pi}{6}\right)$

$=\boxed{\frac{\sqrt{3}}{2}}$

Jan 14-11:19 AM

Evaluate $\lim_{x \to 2} \frac{x^5+4x-40}{x-2} = \frac{2^5+4(2)-40}{2-2}$

$= \frac{32+8-40}{2-2}$

$= \frac{0}{0}$ I.F.

Method I:

Use synthetic div.

2	1	0	0	0	4	-40
		2	4	8	16	40
	1	2	4	8	20	0

$\lim_{x \to 2} \frac{\cancel{(x-2)}(x^4+2x^3+4x^2+8x+20)}{\cancel{x-2}} = 2^4+2(2)^3+4(2)^2+8(2)+20$

$= \boxed{84}$

Method II $\lim_{x \to 2} \frac{x^5+4x-40}{x-2}$

$a=2$

$f(x)=x^5+4x$

$f(2)=40$

$f'(x)=5x^4+4$

$= f'(2)$

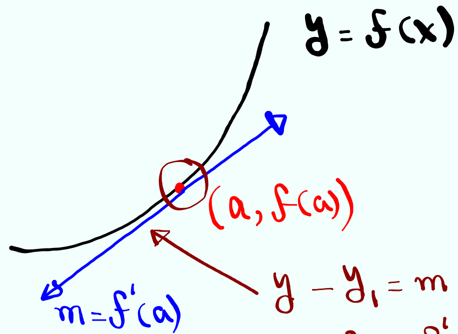
$= 5(2)^4+4$

$= 5 \cdot 16+4$

$= 80+4$

$= \boxed{84}$

Jan 14-11:25 AM



$y - y_1 = m(x-x_1)$

$f(x) - f(a) = f'(a)(x-a)$

$f(x) \approx f(a) + f'(a)(x-a)$

Estimate 2.1^3

Calc. ≈ 9.261

$f(x) = x^3$

$a = 2$

$f'(x) = 3x^2$

$f(2) = 2^3 = 8$ $\quad f'(2) = 12$

$x^3 \approx f(2) + f'(2)(x-2)$

$x^3 = 8 + 12(x-2)$

$2.1^3 = 8 + 12(2.1-2)$

$= 8 + 12(.1)$

$= 8 + 1.2 = \boxed{9.2}$

Jan 14-11:35 AM

Estimate $\sqrt{10}$

Calc. $\sqrt{10} \approx 3.162$

$\sqrt{10} \approx \sqrt{9}$

$a = 9$

$f(x) = \sqrt{x}$

$f(a) = 3$

$f'(x) = \dfrac{1}{2\sqrt{x}}$ $\quad f'(a) = \dfrac{1}{6}$

$f(x) \approx f(a) + f'(a)(x-a)$

$\sqrt{x} \approx 3 + \frac{1}{6}(x-9)$

$\sqrt{10} \approx 3 + \frac{1}{6}(10-9)$

$= 3 + \frac{1}{6} \cdot 1 = \frac{19}{6}$

$\sqrt{10} \approx \frac{19}{6} \approx 3.167$

Jan 14-11:44 AM

Estimate $\tan 46^\circ$

$\tan 46^\circ \approx 1.036$
from Calc

$\tan 46^\circ \approx \tan 45^\circ \approx 1$

$a = 45^\circ = \frac{\pi}{4}$

$f(x) = \tan x$ $\quad f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x$ $\quad f'\left(\frac{\pi}{4}\right) = \sec^2\frac{\pi}{4} = \dfrac{1}{\cos^2\frac{\pi}{4}} = \dfrac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = \dfrac{1}{\frac{2}{4}} = \dfrac{1}{\frac{1}{2}} = \boxed{2}$

$\boxed{f(x) \approx f(a) + f'(a)(x-a)}$

$\tan x \approx 1 + 2(x - 45^\circ)$

$\tan 46^\circ \approx 1 + 2(46^\circ - 45^\circ)$

$= 1 + 2 \cdot 1^\circ = 1 + 2 \cdot \frac{\pi}{180} = 1 + \frac{\pi}{90}$

≈ 1.035

Jan 14-11:49 AM

Class QZ 9

Find $f'(x)$

1) $f(x) = x^3 - 6x^2 + 4x$

$f'(x) = 3x^2 - 6 \cdot 2x + 4 = \boxed{3x^2 - 12x + 4}$

2) $f(x) = \dfrac{x-3}{x+5}$

$f'(x) = \dfrac{1(x+5) - (x-3) \cdot 1}{(x+5)^2}$

$= \dfrac{x+5-x+3}{(x+5)^2} = \boxed{\dfrac{8}{(x+5)^2}}$

Jan 14-11:59 AM